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Eigenfunction expansions and spectral projections for isotropic elasticity outside an obstacle

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Abstract

We consider the operator $-\mathbf{\Delta} - \alpha\, \mathbf{grad}\, \mathrm{div}$ acting on an exterior domain Ω in $\mathbb{R}^n$ (with $\alpha > 0$ and $n = 2, 3$) subject to Dirichlet boundary conditions. The spectral resolution for the operator is written in terms of an expansion of generalized eigenfunctions.

Keywords: Spectral theory; Limiting absorption principle; Generalized eigenfunctions; Elastic plane waves

1. Introduction

In this paper, we consider the self-adjoint operator $\mathbf{L}$ determined by the differential expression

$$-\mathbf{\Delta} - \alpha\, \mathbf{grad}\, \mathrm{div} \tag{1}$$

with $\alpha > 0$ acting on an exterior domain with smooth boundary, and subject to Dirichlet boundary conditions. By an exterior domain, we mean a domain $\Omega \subset \mathbb{R}^n$ given by the unbounded component of the complement of a compact obstacle $\mathcal{O}$. The expression (1)

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arises in the theory of elasticity; for instance, the equation of equilibrium for isotropic bodies is

$$\boldsymbol{\Delta u}(\boldsymbol{x}) + \frac{1}{1-2\sigma}\,\mathbf{grad}\,\operatorname{div}\mathbf{u}(\boldsymbol{x}) = \mathbf{f}(\boldsymbol{x}).$$

The unknown vector field $\mathbf{u}(\boldsymbol{x})$ describes the displacement of each of the points $\boldsymbol{x}$ inside a body after the deformation of this body. The constant σ, known as the *Poisson ratio*, is the ratio of the compression in the transverse direction to the extension in the longitudinal direction for the material; the vector-function $\mathbf{f}(\boldsymbol{x})$ is determined by the internal forces in the body. We refer to [11] and references therein for details on the physical background of this setting.

We deal in this work with the spectral properties of $\mathbf{L}$; our main result is Theorem 5.1. In there, we write the spectral projections of the operator $\mathbf{L}$ in terms of a collection of generalized eigenfunctions associated to the continuous spectrum. The key step in this construction is an expression for the resolvent of $\mathbf{L}$ in terms of this collection of eigenfunctions (Theorem 4.2). The methods used follow, for the main part, those applied by N. Shenk in [16] for the analogous situation for the Laplace operator. Shenk writes the spectral resolution of the Laplace operator on a domain Ω, exterior to a compact C^2 surface with Dirichlet conditions, in terms of an expansion of generalized eigenfunctions. In [16], the spectral projections E_λ for the operator $-\Delta$ acting on Ω are written in the form

$$(E_\lambda f)(x) = \left(\frac{1}{2\pi}\right)^{n/2} \int\limits_{|\xi|^2 \leqslant \lambda} w_+(x,\xi) \int\limits_{\Omega} \overline{w_+(y,\xi)}\, f(y)\, dy\, d\xi, \tag{2}$$

where w_+ are the generalized eigenfunctions known in the literature as the *perturbed plane waves*. These are solutions to the equation

$$(-\Delta w_+)(\cdot,\xi) = \xi^2 w_+(\cdot,\xi) \tag{3}$$

subject to zero (Dirichlet or Neumann) boundary conditions, and such that the functions $w_+(x,\xi) - e^{-ix\cdot\xi}$ satisfy Sommerfeld radiation conditions. In the present work, the role of the perturbed plane waves is played by a collection of vector valued functions which we call the *perturbed elastic plane waves* and denote by $\mathbf{V}_j(\boldsymbol{x},\boldsymbol{p})$ (see Definition 3.1).

In this scheme of things, one has an operator acting on a Hilbert space $\mathcal{H}$ and a collection of generalized eigenfunctions associated to it which determines an integral operator $\mathcal{U}$ (in our example, this is the operator $\mathcal{U}$ defined in (26)). For the operator $\mathcal{U}$ the following two questions naturally arise (see, for instance, [7,16]): the first one, known as *completeness*, is to determine whether $\mathcal{U}^*\mathcal{U}$ equals the identity on $\mathcal{H}$; the second one, known as *orthogonality* is to determine whether the operator $\mathcal{U}$ is onto. Completeness is an immediate consequence of Theorem 5.1; orthogonality is related to the unitary character of the scattering matrix (e.g., [7]), and is one of the deep connections of scattering theory with the expansions of generalized eigenfunctions.

Expansions of generalized eigenfunctions have been used widely in the study of spectral and scattering properties of differential operators with continuous spectrum; this subject dates back to the middle years of the 20th century (we refer to [2,9] for extensive discussions on this). In particular, eigenfunction expansions for partial differential operators on exterior domains have been applied since the nineteen sixties; for example, the Laplace and Schrödinger operators are treated in [6,7,16,17,20], among others. In elasticity, for different settings than ours, eigenfunction expansions techniques related to the continuous spectrum have been developed in [15], and more recently in [3–5,19].

The spectral resolution of the operator (1) acting on an exterior domain with Neumann boundary conditions, is written in terms of an expansion of generalized eigenfunctions in a recent paper by M. Mabrouk and Z. Helali (see [12, Theorem 13]). The class of domains considered by Mabrouk and Helali is wider than ours, and corresponds to domains which satisfy a condition which they call the *elastic local compactness property* (we refer to [12] for the definition). The methods used in [12] are independent to those in the present work. In particular, the radiation conditions considered in the construction of the perturbed (distorted) plane waves are different. Also, the generalized eigenfunctions in [12] are some 3×3 matrices (called the *distorted plane waves* in that reference) that satisfy column-wise equations analogous to (3) for elasticity; as a result of this, the formula in [12] corresponding to (2) involves matrix multiplications. Our choice of the vector fields $\mathbf{V}_j$ as the eigenfunctions gives a formula for the spectral resolution which uses the inner product in $\mathbb{C}^3$ instead; our approach gives an analogue to (2) which, we feel, turns out to be quite natural. Even though the expansions obtained by both approaches should be equivalent, it does not seem to us that this can be verified in an obvious or straightforward manner.

Our work is organized as follows: in Section 2 we define the operator $\mathbf{L}$. In Section 3 we introduce the collection of generalized eigenfunctions. In Section 4 we give the expansions for the resolvent. In Section 5 we finally give the spectral representation in terms of the eigenfunctions.

2. Notation and preliminaries

We will use standard boldface notation to distinguish vectors from scalars. For a generic Hilbert space $\mathcal{H}$, we will denote its internal product by $(\cdot\,,\cdot)_{\mathcal{H}}$, and take it conjugate linear in the second entry. We will write $\bigoplus^n \mathcal{H}$ for the direct sum of n copies of $\mathcal{H}$. The inner product in $\mathbb{C}^n$ will be denoted $\langle\cdot\,,\cdot\rangle$.

Below, we define the operator $\mathbf{L}$.

Definition 2.1. Let $\mathcal{O}$ be a compact obstacle in $\mathbb{R}^n$ with C^∞ boundary, and let Ω be the connected unbounded component of $\mathbb{R}^n \setminus \mathcal{O}$. We define $\mathbf{L}$ as the operator given by the expression 1 acting on Ω, subject to Dirichlet boundary conditions, with domain D given by

$$D = \Big\{\mathbf{F} \in L^2(\Omega; \mathbb{C}^n) \;\Big|\; \lim_{\boldsymbol{x}\to\boldsymbol{x}_0} \mathbf{F}(\boldsymbol{x}) = 0 \text{ for } \boldsymbol{x}_0 \in \partial\Omega \text{ and } \mathbf{L}\,\mathbf{F} \in L^2(\Omega; \mathbb{C}^n)\Big\}.$$

It is well known that the spectrum of $\mathbf{L}$ is absolutely continuous and that it is given by the positive real axis $[0, \infty)$ (e.g., [8,12,18]).

We will denote by $\mathbf{L}_0$ the self-adjoint realization of the expression (1) acting on the whole of $\mathbb{R}^n$.

Throughout this paper, we will restrict ourselves to the case $n = 3$; the results and proofs for the case $n = 2$ are analogous and are given with detail in the author's thesis [13, Section 3.4].

We will denote by $\hat{\mathbf{F}}$ the Fourier transform of $\mathbf{F}$:

$$\hat{\mathbf{F}}(\boldsymbol{p}) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \mathbf{F}(\boldsymbol{x}) e^{-i\langle \boldsymbol{x}, \boldsymbol{p}\rangle}\, d\boldsymbol{p}.$$

For each vector $\boldsymbol{p} = (p_1, p_2, p_3) \in \mathbb{R}^3$ with p_1 and p_2 not simultaneously equal to 0, we will use the notation:

$$\boldsymbol{p}^{\perp} = \frac{|\boldsymbol{p}|}{\sqrt{p_1^2 + p_2^2}} \begin{pmatrix} -p_2 \\ p_1 \\ 0 \end{pmatrix}, \qquad \tilde{\boldsymbol{p}} = \frac{1}{\sqrt{p_1^2 + p_2^2}} \begin{pmatrix} -p_1 p_3 \\ -p_2 p_3 \\ p_1^2 + p_2^2 \end{pmatrix}.$$

We observe that, for each $\boldsymbol{p}$, the vectors $\boldsymbol{p}$, $\boldsymbol{p}^{\perp}$ and $\tilde{\boldsymbol{p}}$ are perpendicular to each other and have the same Euclidean norm.

Elastic waves travel with two different velocities: one corresponding to the *longitudinal waves* and the other corresponding to the *transverse waves* (e.g., [10]); due to this, it will be convenient to introduce the following notation:

$$c_1 = \frac{1}{\sqrt{1+\alpha}}, \qquad c_2 = c_3 = 1.$$

With this, we can introduce the collection of so-called (*unperturbed*) *elastic plane waves* defined for $\boldsymbol{x} \in \mathbb{R}^3$ and $\boldsymbol{p} \in \mathbb{R}^3 \setminus \{(0, 0, p)\}$ by

$$\mathbf{V}_{1,0}(\boldsymbol{x}, \boldsymbol{p}) = e^{ic_1\langle \boldsymbol{x}, \boldsymbol{p}\rangle} \frac{\boldsymbol{p}}{|\boldsymbol{p}|}, \qquad \mathbf{V}_{2,0}(\boldsymbol{x}, \boldsymbol{p}) = e^{ic_2\langle \boldsymbol{x}, \boldsymbol{p}\rangle} \frac{\boldsymbol{p}^{\perp}}{|\boldsymbol{p}|},$$
$$\mathbf{V}_{3,0}(\boldsymbol{x}, \boldsymbol{p}) = e^{ic_3\langle \boldsymbol{x}, \boldsymbol{p}\rangle} \frac{\tilde{\boldsymbol{p}}}{|\boldsymbol{p}|}.$$

It can be verified (e.g., [13, Section 3.1]) that these vector fields are formal solutions to the equation

$$\mathbf{L}\,\mathbf{V}_{j,0}(\cdot, \boldsymbol{p}) = |\boldsymbol{p}|^2\, \mathbf{V}_{j,0}(\cdot, \boldsymbol{p}).$$

We note that $\mathbf{V}_{j,0}(\boldsymbol{x}, \cdot)$ can be extended continuously to $\mathbb{R}^3 \setminus \mathbf{0}$ only for $j = 1$. However, all these elastic plane waves are defined and continuous outside a zero-measure subset of $\mathbb{R}^3 \times \mathbb{R}^3$, and $|\mathbf{V}_{j,0}(\boldsymbol{x}, \boldsymbol{p})| = 1$.

Definition 2.2. Let $s \in \mathbb{C}$. We say that a vector field $\mathbf{u}$ satisfies outgoing s-elastic radiation conditions, if it can be decomposed as the sum of a gradient component $\mathbf{u}^{(g)}$ plus a curl component $\mathbf{u}^{(c)}$, both of order $O(1/r)$ as $r \to \infty$, which satisfy

$$\begin{cases} \dfrac{\partial \mathbf{u}^{(g)}}{\partial r}(\boldsymbol{x}) - ic_1 s \mathbf{u}^{(g)}(\boldsymbol{x}) = o\left(\dfrac{1}{r}\right), \\[2ex] \dfrac{\partial \mathbf{u}^{(c)}}{\partial r}(\boldsymbol{x}) - is \mathbf{u}^{(c)}(\boldsymbol{x}) = o\left(\dfrac{1}{r}\right), \end{cases} \qquad r \to \infty. \tag{4}$$

We say that $\mathbf{u}$ satisfies outgoing s-elastic radiation conditions if the same statement holds true with the sign '$-$' replaced by '$+$' in (4).

Note. A definition of radiation conditions for general hypo-elliptic operators and systems is given in [22]. Definition 2.2 coincides with it in our particular case.

3. The generalized eigenfunctions

Let $\mathbf{H}$ be a continuous vector field defined on the boundary $\partial\Omega$, and let $s \in \mathbb{C}$. It is known (e.g., [10]) that there exists a unique solution to the problem

$$\begin{cases} [\mathbf{L} - s^2]\mathbf{u}(\boldsymbol{x}) = \mathbf{0}, & \boldsymbol{x} \in \Omega, \\ \mathbf{u}(\boldsymbol{x}) = \mathbf{H}(\boldsymbol{x}), & \boldsymbol{x} \in \partial\Omega, \end{cases} \tag{5}$$

satisfying s-outgoing elastic radiation conditions. The same statement is true for the incoming conditions.

Taking this into account, we define

$$\mathbf{W}_j(\boldsymbol{x}, \boldsymbol{p}; s) = \mathbf{V}_{j,0}(\boldsymbol{x}, \boldsymbol{p}) + \mathbf{u}_j(\boldsymbol{x}, \boldsymbol{p}; s),$$

where $\mathbf{u}_j(\cdot, \boldsymbol{p}; s)$ satisfies s-outgoing elastic radiation conditions and is solution to the problem (5) with $\mathbf{H}(\cdot) = -\mathbf{V}_{j,0}(\cdot, \boldsymbol{p})$.

Estimates for $\mathbf{u}_j(\boldsymbol{x}, \boldsymbol{p}; s)$ can be given (see, for example, [22, Theorem 1 in Chapter VIII]); in particular, for each fixed x and s, the vector fields $\mathbf{W}_j(\boldsymbol{x}, \cdot; s)$ are in $L^2_{\mathrm{loc}}(\mathbb{R}^3; \mathbb{C}^3)$.

It will be convenient to consider the vector fields $\mathbf{W}_j(\boldsymbol{x}, \boldsymbol{p}; s)$ as defined for all $\boldsymbol{x} \in \mathbb{R}^3$ by setting

$$\mathbf{W}_j(\boldsymbol{x}, \boldsymbol{p}; s) = 0, \quad \text{for all } \boldsymbol{x} \notin \Omega.$$

By definition, the vector fields $\mathbf{W}_j(\boldsymbol{x}, \boldsymbol{p}; s)$ satisfy formally the equation

$$(\mathbf{L} - s^2)\mathbf{W}_j(\boldsymbol{x}, \boldsymbol{p}; s) = (|p|^2 - s^2)\mathbf{V}_{j,0}(\boldsymbol{x}, \boldsymbol{p}), \quad x \in \Omega,$$

and homogeneous Dirichlet boundary conditions in $\partial\Omega$.

Definition 3.1. Let $\mathbf{V}_j(\boldsymbol{x}, \boldsymbol{p})$ be given by $\mathbf{V}_j(\boldsymbol{x}, \boldsymbol{p}) = \mathbf{W}_j(\boldsymbol{x}, \boldsymbol{p}; |\boldsymbol{p}|)$. We call these vector fields the perturbed elastic plane waves.

The elastic plane waves defined above are solutions to the problem (5) with $\mathbf{H} = 0$.

The vector fields $\mathbf{W}_j(\boldsymbol{x}, \boldsymbol{p}; s)$, and in particular the elastic plane waves, can be written in the form (e.g., [10, Theorem VI.13]; see also Sections II.1, II.2 and I.2 in the same reference):

$$\mathbf{W}_j(\boldsymbol{x}, \boldsymbol{p}; s) = \mathbf{V}_j(\boldsymbol{x}, \boldsymbol{p}) + \frac{1}{(2\pi)^{\frac{n-1}{2}}} \int_{\partial\Omega} \mathsf{A}(\boldsymbol{x}, \boldsymbol{y}; s)\boldsymbol{\varphi}_j(\boldsymbol{y}, \boldsymbol{p}; s)\, d\boldsymbol{y}. \tag{6}$$

Here, $\boldsymbol{\varphi}_j$ is a density vector valued function, and A is the 3×3 Green matrix defined as follows:

For a vector field $\mathbf{u}$ defined on a surface S, let Γ_j and $\mathcal{T}$ be given by

$$\Gamma_j(\boldsymbol{x},\boldsymbol{y};s) = -\,\mathbf{grad}\,\mathrm{div}\left(\frac{e^{ic_1 r}}{r}\right) + \mathbf{curl}\,\mathbf{curl}\left(\frac{e^{ir}}{r}\right),$$

$$\mathcal{T}\mathbf{u} = -2\frac{\partial \mathbf{u}}{\partial n} + (\alpha+1)\boldsymbol{n}\,\mathrm{div}\,\mathbf{u} - (\boldsymbol{n}\times\mathbf{curl}\,\mathbf{u}),$$

where $\boldsymbol{n}$ is the vector normal to S, pointing outwards; then, the jth row of A is defined by $\mathcal{T}\Gamma_j$. The operator $\mathcal{T}$ is known as the *stress operator*. The vector fields given by the columns of the matrix A are solutions—with respect to either variable—of the equation

$$\left(\mathbf{L} - s^2\right)\mathsf{A}^{(j)}(\boldsymbol{x},\boldsymbol{y};s) = \delta(\boldsymbol{x}-\boldsymbol{y}).$$

One consequence of (6) is that the $\mathbf{W}_j$ are of the form

$$\mathbf{W}_j(\boldsymbol{x},\boldsymbol{p};s) = \mathbf{V}_{j,0}(\boldsymbol{x},\boldsymbol{p}) + \mathbf{c}_j\left(\frac{\boldsymbol{x}}{r},\frac{\boldsymbol{p}}{|\boldsymbol{p}|}\right)\frac{e^{ic_1 r}}{r} + \mathbf{d}_j\left(\frac{\boldsymbol{x}}{r},\frac{\boldsymbol{p}}{|\boldsymbol{p}|}\right)\frac{e^{ic_2 r}}{r} + \mathbf{g}_s(\boldsymbol{x},\boldsymbol{p}), \tag{7}$$

where $\mathbf{c}_j$ and $\mathbf{d}_j$ are of class C^∞ and for every fixed $\boldsymbol{p}$ the vector valued function $\mathbf{g}_s(\boldsymbol{x},\boldsymbol{p})$, together with its partial derivatives of all order, are $O(r^{-2})$. Since $\mathbf{L} - s^2$ is elliptic, this implies that, for every $m \in \mathbb{N}$, $\mathbf{g}_s(\cdot,\boldsymbol{p})$ belongs to the Sobolev space $H^m(\Omega;\mathbb{C}^n)$ (e.g., [1, Theorem 10.7]).

4. Expansions for the resolvent

In this section we will express, in terms of the vector fields $\mathbf{W}_j$ and $\mathbf{V}_j$ defined above, the resolvent of $\mathbf{L}$ at $s \in \mathbb{C}$ with $\mathrm{Im}\, s > 0$; this is Theorem 4.2. We will also obtain an expression for the norm of the resolvent (Proposition 4.4) which will be useful in the next section.

We first prove the following technical lemma.

Lemma 4.1. *For* $\mathbf{F} \in L^2(\mathbb{R}^3;\mathbb{C}^3)$ *define*

$$\mathbf{A}^{(1)}_{\mathbf{F}}(\boldsymbol{p}) = \boldsymbol{p}\langle\boldsymbol{p},\hat{\mathbf{F}}(\boldsymbol{p})\rangle, \qquad \mathbf{A}^{(2)}_{\mathbf{F}}(\boldsymbol{p}) = \boldsymbol{p}^\perp\langle\boldsymbol{p}^\perp,\hat{\mathbf{F}}(\boldsymbol{p})\rangle, \qquad \mathbf{A}^{(3)}_{\mathbf{F}}(\boldsymbol{p}) = \tilde{\boldsymbol{p}}\langle\tilde{\boldsymbol{p}},\hat{\mathbf{F}}(\boldsymbol{p})\rangle.$$

Then

$$\mathbf{A}^{(1)}_{\mathbf{F}}(\boldsymbol{p}) + \mathbf{A}^{(2)}_{\mathbf{F}}(\boldsymbol{p}) + \mathbf{A}^{(3)}_{\mathbf{F}}(\boldsymbol{p}) = \widehat{-\boldsymbol{\Delta}\mathbf{F}}(\boldsymbol{p}).$$

Proof. We first note that

$$\mathbf{A}^{(1)}_{\mathbf{F}}(\boldsymbol{p}) = \boldsymbol{p}\left(p_1\hat{F}_1(\boldsymbol{p}) + p_2\hat{F}_2(\boldsymbol{p}) + p_3\hat{F}_3(\boldsymbol{p})\right) = i\,\boldsymbol{p}\,\widehat{\mathrm{div}\,\mathbf{F}}(\boldsymbol{p}) = -\widehat{\mathbf{grad}\,\mathrm{div}\,\mathbf{F}}(\boldsymbol{p}).$$

In a similar way, for $\boldsymbol{p} \neq (0,0,p)$ we have:

$$\mathbf{A}_{\mathbf{F}}^{(2)}(\boldsymbol{p}) = \frac{|\boldsymbol{p}|^2}{p_1^2 + p_2^2} \begin{pmatrix} p_2^2 & -p_1 p_2 & 0 \\ -p_1 p_2 & p_1^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \hat{F}_1(\boldsymbol{p}) \\ \hat{F}_2(\boldsymbol{p}) \\ \hat{F}_3(\boldsymbol{p}) \end{pmatrix},$$

$$\mathbf{A}_{\mathbf{F}}^{(3)}(\boldsymbol{p}) = \frac{1}{p_1^2 + p_2^2} \begin{pmatrix} p_1^2 p_3^2 & p_1 p_2 p_3^2 & -p_1 p_3 (p_1^2 + p_2^2) \\ p_1 p_2 p_3^2 & p_2^2 p_3^2 & -p_2 p_3 (p_1^2 + p_2^2) \\ -p_1 p_3 & -p_2 p_3 & (p_1^2 + p_2^2)^2 \end{pmatrix} \begin{pmatrix} \hat{F}_1(\boldsymbol{p}) \\ \hat{F}_2(\boldsymbol{p}) \\ \hat{F}_3(\boldsymbol{p}) \end{pmatrix}.$$

Summing up these matrices yields the equality

$$\begin{aligned} \mathbf{A}_{\mathbf{F}}^{(2)}(\boldsymbol{p}) + \mathbf{A}_{\mathbf{F}}^{(3)}(\boldsymbol{p}) &= -\begin{pmatrix} p_2^2 + p_3^2 & -p_1 p_2 & -p_1 p_3 \\ -p_1 p_2 & p_1^2 + p_3^2 & -p_2 p_3 \\ -p_1 p_3 & -p_2 p_3 & p_1^2 + p_2^2 \end{pmatrix} \begin{pmatrix} \hat{\mathbf{F}}_1(\boldsymbol{p}) \\ \hat{\mathbf{F}}_2(\boldsymbol{p}) \\ \hat{\mathbf{F}}_3(\boldsymbol{p}) \end{pmatrix} \\ &= \widehat{\mathbf{curl}\,\mathbf{curl}}\,\mathbf{F}(\boldsymbol{p}). \end{aligned} \tag{8}$$

The lemma then follows from the identity (see, e.g., [14]):

$$\boldsymbol{\Delta} = \mathbf{grad}\,\mathrm{div} - \mathbf{curl}\,\mathbf{curl}. \qquad \square$$

Theorem 4.2. *Let s be any complex number with positive imaginary part, and let $\mathbf{F} \in C_0^\infty(\Omega; \mathbb{C}^3)$. Then, the resolvent of the operator $\mathbf{L}$ at the point s^2 is given by*

$$\big(R_L(s^2)\mathbf{F}\big)(\boldsymbol{x}) = \sum_{j=1}^{3} \left(\frac{c_j}{2\pi}\right)^3 \int_{\mathbb{R}^3} \frac{\mathbf{W}_j(\boldsymbol{x}, \boldsymbol{p}; s)}{|p|^2 - s^2} \int_{\Omega} \big\langle \mathbf{F}(\boldsymbol{y}), \mathbf{V}_{j,0}(\boldsymbol{y}, \boldsymbol{p}) \big\rangle \, d\boldsymbol{y}\, d\boldsymbol{p}. \tag{9}$$

If $\mathbf{H} \in C_0^\infty(\operatorname{int} \mathcal{O}; \mathbb{C}^3)$, then

$$\int_{\mathbb{R}^3} \frac{\mathbf{W}_j(\boldsymbol{x}, \boldsymbol{p}; s)}{|p|^2 - s^2} \int_{\mathbb{R}^3} \big\langle \mathbf{H}(\boldsymbol{y}), \mathbf{V}_{j,0}(\boldsymbol{y}, \boldsymbol{p}) \big\rangle \, d\boldsymbol{y}\, d\boldsymbol{p} = 0, \quad \textit{for each } j = 1, 2, 3. \tag{10}$$

Proof. From the definitions of $\mathbf{A}_{\mathbf{F}}^{(j)}$ and $\mathbf{V}_{j,0}$, the following relation holds:

$$\mathbf{V}_{j,0}(\boldsymbol{x}, \boldsymbol{p}) \int_{\Omega} \big\langle \mathbf{F}(\boldsymbol{y}), \mathbf{V}_{j,0}(\boldsymbol{y}, \boldsymbol{p}) \big\rangle \, d\boldsymbol{y} = \frac{(2\pi)^{3/2} e^{i c_j \langle \boldsymbol{x}, \boldsymbol{p} \rangle}}{c_j^2 |\boldsymbol{p}|^2} \mathbf{A}_{\mathbf{F}}^{(j)}(c_j \boldsymbol{p}).$$

Since $\mathbf{F} \in C_0^\infty(\operatorname{int} \mathcal{O}; \mathbb{C}^3)$ and $\mathbf{V}_{j,0}(\boldsymbol{x}, \boldsymbol{p})$ is bounded we can integrate over $\boldsymbol{p}$, in order to obtain:

$$\int_{\mathbb{R}^3} \mathbf{V}_{j,0}(\boldsymbol{x}, \boldsymbol{p}) \int_{\Omega} \big\langle \mathbf{F}(\boldsymbol{y}), \mathbf{V}_{j,0}(\boldsymbol{y}, \boldsymbol{p}) \big\rangle \, d\boldsymbol{y}\, d\boldsymbol{p} = \frac{(2\pi)^{3/2}}{c_j^3} \int_{\mathbb{R}^3} \frac{e^{i \langle \boldsymbol{x}, \boldsymbol{p} \rangle}}{|\boldsymbol{p}|^2} \mathbf{A}_{\mathbf{F}}^{(j)}(\boldsymbol{p})\, d\boldsymbol{p}.$$

From this and Lemma 4.1, it follows that

$$\sum_{j=1}^{3}\left(\frac{c_j}{2\pi}\right)^3 \int\limits_{\mathbb{R}^3} \mathbf{V}_{j,0}(\boldsymbol{x},\boldsymbol{p}) \int\limits_{\Omega} \langle \mathbf{F}(\boldsymbol{y}), \mathbf{V}_{j,0}(\boldsymbol{y},\boldsymbol{p})\rangle \, d\boldsymbol{y}\, d\boldsymbol{p}$$

$$= -\frac{1}{(2\pi)^{3/2}} \int\limits_{\mathbb{R}^3} \widehat{\boldsymbol{\Delta}\mathbf{F}}(\boldsymbol{p}) \frac{e^{i\langle \boldsymbol{x},\boldsymbol{p}\rangle}}{|\boldsymbol{p}|^2}\, d\boldsymbol{p} = -\frac{1}{(2\pi)^{3/2}} \int\limits_{\mathbb{R}^3} \hat{\mathbf{F}}(\boldsymbol{p}) e^{i\langle \boldsymbol{x},\boldsymbol{p}\rangle}\, d\boldsymbol{p} = \mathbf{F}(\boldsymbol{x}). \tag{11}$$

On the other hand, the expression in the right-hand side of (9) makes sense, because $\mathbf{F} \in C_0^\infty(\Omega;\mathbb{C}^3)$ and $\mathbf{W}_j(\boldsymbol{x},\cdot\,;s) \in L^2_{\mathrm{loc}}(\mathbb{R}^3;\mathbb{C}^3)$. Also, since $\mathbf{F} \in C_0^\infty(\Omega;\mathbb{C}^3)$ and $\mathbf{W}_j(\boldsymbol{x},\boldsymbol{p};s)$ vanishes for $\boldsymbol{x} \in \partial\Omega$, that expression lies in the domain of $\mathbf{L}$; applying then $\mathbf{L} - s^2$ to (9) yields, for $\boldsymbol{x} \in \Omega$, the left-hand side of (11) and equality (9) follows.

To prove (10), we first note that the expression (11) remains true if we replace $\mathbf{F}$ by $\mathbf{H}$. Then, the left-hand side of (10) vanishes if we apply $\mathbf{L}_0 - s^2$ to it; but this gives the desired result, since $s^2 \notin \mathbb{R}$ cannot be an eigenvalue of the self-adjoint operator $\mathbf{L}$ acting in the obstacle. □

We will now introduce an operator $\mathcal{U}_0 : L^2(\mathbb{R}^3;\mathbb{C}^3) \to \bigoplus^3 L^2(\mathbb{R}^3)$, that will play in this work the role played by the Fourier transform in [16]:

$$(\mathcal{U}_0\mathbf{F})_1(\boldsymbol{p}) = \frac{c_1^{3/2}}{|\boldsymbol{p}|}\langle \hat{\mathbf{F}}(c_1\boldsymbol{p}), \boldsymbol{p}\rangle, \qquad (\mathcal{U}_0\mathbf{F})_2(\boldsymbol{p}) = \frac{1}{|\boldsymbol{p}|}\langle \hat{\mathbf{F}}(\boldsymbol{p}), \boldsymbol{p}^\perp\rangle$$

$$(\mathcal{U}_0\mathbf{F})_3(\boldsymbol{p}) = \frac{1}{|\boldsymbol{p}|}\langle \hat{\mathbf{F}}(\boldsymbol{p}), \tilde{\boldsymbol{p}}\rangle.$$

Theorem 4.3. *$\mathcal{U}_0$ is a unitary operator from $L^2(\mathbb{R}^3;\mathbb{C}^3)$ onto $\bigoplus^3 L^2(\mathbb{R}^3)$.*

Proof. First we consider an arbitrary $f \in C_0^\infty(\mathbb{R}^3)$ and note that

$$(\mathcal{U}_0\,\mathbf{grad}\, f)_1(\boldsymbol{p}) = \frac{c_1^{3/2}}{|\boldsymbol{p}|}\langle \widehat{\mathbf{grad}\, f}(c_1\boldsymbol{p}), \boldsymbol{p}\rangle = \frac{i c_1^{5/2} \hat{f}(c_1|\boldsymbol{p}|)}{|\boldsymbol{p}|}\langle \boldsymbol{p}, \boldsymbol{p}\rangle$$
$$= i c_1^{5/2} |\boldsymbol{p}| \hat{f}(c_1|\boldsymbol{p}|). \tag{12}$$

In a similar way, the equalities

$$(\mathcal{U}_0\,\mathbf{grad}\, f)_2(\boldsymbol{p}) = (\mathcal{U}_0\,\mathbf{grad}\, f)_3(\boldsymbol{p}) = 0, \tag{13}$$

follow from the fact that $\boldsymbol{p}^\perp$ and $\tilde{\boldsymbol{p}}$ are both perpendicular to $\boldsymbol{p}$.

On the other hand, we have for any vector field $\mathbf{F} \in C_0^\infty(\mathbb{R}^3;\mathbb{C}^3)$:

$$(\mathcal{U}_0\,\mathbf{curl}\,\mathbf{F})_1(\boldsymbol{p}) = \frac{1}{\sqrt{c_1}|\boldsymbol{p}|}\langle \widehat{\mathbf{curl}\,\mathbf{F}}(c_1\boldsymbol{p}), \boldsymbol{p}\rangle$$
$$= \frac{i\sqrt{c_1}}{|\boldsymbol{p}|}\begin{pmatrix} p_1 & p_2 & p_3 \end{pmatrix} \begin{pmatrix} 0 & -p_3 & p_2 \\ p_3 & 0 & -p_1 \\ -p_2 & p_1 & 0 \end{pmatrix} \begin{pmatrix} \hat{F}_1(\boldsymbol{p}) \\ \hat{F}_2(\boldsymbol{p}) \\ \hat{F}_3(\boldsymbol{p}) \end{pmatrix} = 0.$$

Now, if we take $\mathbf{G} = (\partial_2 f, -\partial_1 f, 0) = \mathbf{curl}(0,0,f)$, we obtain for $\boldsymbol{p} \neq (0,0,p)$:

$$
\begin{aligned}
&(\mathcal{U}_0\,\mathbf{curl}\,\mathbf{G})_2(\boldsymbol{p}) \\
&\quad = \frac{1}{|\boldsymbol{p}|^2}\bigl\langle \widehat{\mathbf{curl}\,\mathbf{G}}(\boldsymbol{p}), \boldsymbol{p}^{\perp}\bigr\rangle \\
&\quad = \frac{-1}{|\boldsymbol{p}|\sqrt{p_1^2+p_2^2}}\,(-p_2 \quad p_1 \quad 0)\begin{pmatrix} 0 & -p_3 & p_2 \\ p_3 & 0 & -p_1 \\ -p_2 & p_1 & 0 \end{pmatrix}\begin{pmatrix} p_2 \hat{f}(\boldsymbol{p}) \\ -p_1 \hat{f}(\boldsymbol{p}) \\ 0 \end{pmatrix} = 0.
\end{aligned}
$$

Just in the same way we can derive, for $\boldsymbol{p} \neq (0,0,p)$, the relation

$$(\mathcal{U}_0\,\mathbf{curl}\,\mathbf{G})_3(\boldsymbol{p}) = \sqrt{p_1^2+p_2^2}\,|\boldsymbol{p}|\,\hat{f}(\boldsymbol{p}).$$

Proceeding just like above with the vector field $\mathbf{G}$, it can be seen that for $\boldsymbol{p} \neq (0,0,p)$:

$$(\mathcal{U}_0\,\mathbf{G})_2(\boldsymbol{p}) = -i\sqrt{p_1^2+p_2^2}\,\hat{f}(\boldsymbol{p}), \qquad (\mathcal{U}_0\,\mathbf{G})_3(\boldsymbol{p}) = 0.$$

For $\boldsymbol{p} = (0,0,p)$, since for this $\boldsymbol{p}$ we have that $\hat{\mathbf{G}}(\boldsymbol{p}) = \mathbf{0}$, the equalities $(\mathcal{U}_0\,\mathbf{curl}\,\mathbf{G})_j(\boldsymbol{p}) = (\mathcal{U}_0\mathbf{G})_j(\boldsymbol{p}) = 0$ hold true.

From all this discussion we can conclude that $\mathcal{U}_0$ has dense range in $\bigoplus^3 L^2(\mathbb{R}^3)$.

Now, due to relations 12 and 13, proving that $\mathcal{U}_0$ maps the vector field $\mathbf{grad}\,f$ isometrically into $\bigoplus^3 L^2(\mathbb{R}^3)$ reduces to verify that

$$c_1^5 \int_{\mathbb{R}^3} |\boldsymbol{p}|^2 \bigl|\hat{f}(c_1\boldsymbol{p})\bigr|^2\,d\boldsymbol{p} = \int_{\mathbb{R}^3} \bigl\langle \mathbf{grad}\,f(\boldsymbol{x}), \mathbf{grad}\,f(\boldsymbol{x})\bigr\rangle\,d\boldsymbol{x}. \tag{14}$$

By the divergence theorem and the unitarity of the Fourier transform on $L^2(\mathbb{R}^3)$, we have:

$$\int_{\mathbb{R}^3} \bigl\langle \mathbf{grad}\,f(\boldsymbol{x}), \mathbf{grad}\,f(\boldsymbol{x})\bigr\rangle\,d\boldsymbol{x} = -\int_{\mathbb{R}^3} f(\boldsymbol{x})\Delta f(\boldsymbol{x})\,d\boldsymbol{x} = -\int_{\mathbb{R}^3} \hat{f}(\boldsymbol{p})\widehat{\Delta f}(\boldsymbol{p})\,d\boldsymbol{p}$$

and (14) follows.

Similarly, for $\mathbf{G}$ as above:

$$
\begin{aligned}
\|\mathbf{G}\|^2_{L^2(\mathbb{R}^3;\mathbb{C}^3)} &= \|\partial_2 f\|^2_{L^2(\mathbb{R}^3)} + \|\partial_1 f\|^2_{L^2(\mathbb{R}^3)} = \bigl\|\widehat{\partial_2 f}\bigr\|^2_{L^2(\mathbb{R}^3)} + \bigl\|\widehat{\partial_1 f}\bigr\|^2_{L^2(\mathbb{R}^3)} \\
&= \int_{\mathbb{R}^3} \bigl(p_1^2+p_2^2\bigr)\bigl|\hat{f}(\boldsymbol{p})\bigr|^2\,d\boldsymbol{p} = \|\mathcal{U}_0\mathbf{G}\|^2_{\bigoplus^3 L^2(\mathbb{R}^3)}.
\end{aligned}
$$

Finally, for the vector field $\mathbf{curl}\,\mathbf{G} = \mathbf{curl}\,\mathbf{curl}(0,0,f)$, we have:

$$
\begin{aligned}
\|\,\mathbf{curl}\,\mathbf{G}\|^2_{L^2(\mathbb{R}^3;\mathbb{C}^3)} &= \bigl\|\widehat{\partial_2\partial_3 f}\bigr\|^2_{L^2(\mathbb{R}^3)} + \bigl\|\widehat{\partial_1\partial_3 f}\bigr\|^2_{L^2(\mathbb{R}^3)} + \bigl\|\widehat{\partial_{22} f} + \widehat{\partial_{33} f}\bigr\|^2_{L^2(\mathbb{R}^3)} \\
&= \int_{\mathbb{R}^3} \bigl(p_1^2+p_2^2\bigr)|\boldsymbol{p}|^2\bigl|\hat{f}(\boldsymbol{p})\bigr|^2\,d\boldsymbol{p} = \|\mathcal{U}_0\,\mathbf{curl}\,\mathbf{G}\|^2_{\bigoplus^3 L^2(\mathbb{R}^3)}.
\end{aligned}
$$

The collection of vector fields

$$\left\{\mathbf{grad}\, f \;\middle|\; f \in C_0^\infty(\mathbb{R}^3)\right\} \cup \left\{\mathbf{curl}\,\mathbf{F} \;\middle|\; \mathbf{F} \in C_0^\infty(\mathbb{R}^3; \mathbb{C}^3)\right\}$$

is dense in $L^2(\mathbb{R}^3; \mathbb{C}^3)$ (e.g., [23]), so that the map $\mathcal{U}_0$ extends isometrically to the whole of $L^2(\mathbb{R}^3; \mathbb{C}^3)$. As we have already proved that it has dense range, it must be unitary. □

In the proposition below, we use the operator $\mathcal{U}_0$ to obtain an expression for the norm of the resolvent of $\mathbf{L}$.

Proposition 4.4. *For* $\mathbf{F} \in C_0^\infty(\Omega; \mathbb{C}^3)$ *and* $\operatorname{Im} s > 0$, *let*

$$B_{\mathbf{F}}^{(j)}(\boldsymbol{p}; s) = \left(\frac{c_1}{2\pi}\right)^{3/2} \int_\Omega \frac{\langle \mathbf{F}(\boldsymbol{x}), \mathbf{W}_j(\boldsymbol{x}, \boldsymbol{p}; s)\rangle}{|\boldsymbol{p}|^2 - s^2}\, d\boldsymbol{x}.$$

Then, the norm of the resolvent of $\mathbf{L}$ *at the point* $\bar{s}^{\,2}$ *is given by*

$$\left\| R_{\mathbf{L}}\left(\bar{s}^{\,2}\right)\mathbf{F}\right\|^2_{L^2(\Omega;\mathbb{C}^3)} = \sum_{j=1}^{3} \left\| B_{\mathbf{F}}^{(j)}(\cdot\,; s)\right\|_{L^2(\mathbb{R}^3)}.$$

Proof. For $\mathbf{F}$ and $\mathbf{G}$ in $C_0^\infty(\Omega; \mathbb{C}^3)$,

$$\left(\mathbf{F}, R_{\mathbf{L}}\left(s^2\right)\mathbf{G}\right)_{L^2(\Omega;\mathbb{C}^3)} = \sum_{k=1}^{3} \int_\Omega F_k(\boldsymbol{x}) \left[R_{\mathbf{L}}\left(s^2\right)\mathbf{G}\right]_k(\boldsymbol{x})\, d\boldsymbol{x}. \tag{15}$$

Denote the kth component of $\mathbf{W}_j$ with respect to rectangular coordinates by $W_{j,k}$. Theorem 4.2 implies that the kth component of $R_{\mathbf{L}}(s^2)\mathbf{G}$ is

$$\begin{aligned}\left[R_{\mathbf{L}}\left(s^2\right)\mathbf{G}\right]_k(\boldsymbol{x}) &= \sum_{j=1}^{3} \left(\frac{c_j}{2\pi}\right)^3 \int_{\mathbb{R}^3} \frac{W_{j,k}(\boldsymbol{x}, \boldsymbol{p}; s)}{|\boldsymbol{p}|^2 - s^2} \int_\Omega \left\langle \mathbf{G}(\boldsymbol{y}), \mathbf{V}_{j,0}(\boldsymbol{y}, \boldsymbol{p})\right\rangle d\boldsymbol{y}\, d\boldsymbol{p} \\ &= \sum_{j=1}^{3} \left(\frac{c_j}{2\pi}\right)^{3/2} \int_{\mathbb{R}^3} \frac{W_{j,k}(\boldsymbol{x}, \boldsymbol{p}; s)}{|\boldsymbol{p}|^2 - s^2} [\mathcal{U}_0\mathbf{G}']_j(\boldsymbol{p})\, d\boldsymbol{p}\end{aligned}$$

where $\mathbf{G}'$ is the extension of $\mathbf{G}$ to the whole of $\mathbb{R}^3$ made by taking it to be identically 0 in the obstacle.

Since $\mathbf{G}' \in C_0^\infty(\mathbb{R}^3; \mathbb{C}^3)$, we have that each $[\mathcal{U}_0\mathbf{G}']_k$ is in $\mathsf{S}(\mathbb{R}^3)$, the Schwarz class of C^∞-smooth functions of rapid decrease; we can then substitute the above equality into the right-hand side of (15) and interchange the order of integration, to obtain:

$$\begin{aligned}\left(\mathbf{F}, R_{\mathbf{L}}\left(s^2\right)\mathbf{G}\right)_{L^2(\Omega;\mathbb{C}^3)} &= \sum_{j,k=1}^{3} \left(\frac{c_j}{2\pi}\right)^{3/2} \int_{\mathbb{R}^3}\int_\Omega F_k(\boldsymbol{x}) \frac{W_{j,k}(\boldsymbol{x}, \boldsymbol{p}; s)}{|\boldsymbol{p}|^2 - s^2} [\mathcal{U}_0\mathbf{G}']_j(\boldsymbol{p})\, d\boldsymbol{x}\, d\boldsymbol{p} \\ &= \sum_{j=1}^{3} \left(\frac{c_j}{2\pi}\right)^{3/2} \int_{\mathbb{R}^3} [\mathcal{U}_0\mathbf{G}']_j(\boldsymbol{p}) \int_\Omega \left\langle \mathbf{F}(\boldsymbol{x}), \mathbf{W}_j(\boldsymbol{x}, \boldsymbol{p}; s)\right\rangle d\boldsymbol{x}\, d\boldsymbol{p}\end{aligned}$$

$$= \sum_{j=1}^{3} \int_{\mathbb{R}^3} B_{\mathbf{F}}^{(j)}(\boldsymbol{p}; s)[\mathcal{U}_0 \mathbf{G}]_j \, d\boldsymbol{p}$$
$$= \left(\mathsf{B}_{\mathbf{F}}(\cdot\, s), \overline{\mathcal{U}_0 \mathbf{G}}\right)_{\bigoplus^3 L^2(\mathbb{R}^3)}$$

where, for the sake of briefness, we have written

$$\mathsf{B}_{\mathbf{F}}(\cdot\,; s) = \left\{B_{\mathbf{F}}^{(1)}(\cdot\,; s), B_{\mathbf{F}}^{(2)}(\cdot\,; s), B_{\mathbf{F}}^{(3)}(\cdot\,; s)\right\} \in \bigoplus^{3} L^2\left(\mathbb{R}^3\right). \tag{16}$$

We extend $R_{\mathbf{L}}(\bar{s}^{\,2})\mathbf{F}$ to the whole of $\mathbb{R}^3$ by defining it to be identically 0 inside the obstacle; we can then write the identity

$$\left(R_{\mathbf{L}}\left(\bar{s}^2\right)\mathbf{F}, \mathbf{G}'\right)_{L^2(\mathbb{R}^3;\mathbb{C}^3)} = \left(\mathsf{B}_{\mathbf{F}}(\cdot\,; s), \overline{\mathcal{U}_0 \mathbf{G}'}\right)_{\bigoplus^3 L^2(\mathbb{R}^3)}. \tag{17}$$

For $\mathbf{H} \in C_0^\infty(\operatorname{int}\mathcal{O}; \mathbb{C}^3)$, in a similar way as above and by means of (10), we can see that

$$\left(\mathsf{B}_{\mathbf{F}}(\cdot\,; s), \overline{\mathcal{U}_0 \mathbf{G}}\right)_{\bigoplus^3 L^2(\mathbb{R}^3)} = 0.$$

So, formula (17) is also valid if we replace $\mathbf{G}'$ with $\mathbf{H}$.

Since $\mathcal{U}_0$ is unitary, this means that

$$\mathcal{U}_0 \, R_{\mathbf{L}}\left(\bar{s}^2\right)\mathbf{F} = \overline{\mathsf{B}_{\mathbf{F}}(\cdot\,; s)} \tag{18}$$

and the proposition follows. □

5. The spectral projections

Now we are ready to state and prove our main result.

Theorem 5.1. *Let $\lambda > 0$. For $t \geqslant 0$, denote by $B(t)$ the ball of radius t and center at the origin. The spectral projections $E_\lambda(\mathbf{L})$ are given by*

$$\left(E_\lambda(\mathbf{L})\mathbf{F}\right)(\boldsymbol{x}) = \sum_{j=1}^{3} \left(\frac{c_j}{2\pi}\right)^3 \int_{B(\sqrt{\lambda})} \mathbf{V}_j(\boldsymbol{x}; \boldsymbol{p}) \int_{\Omega} \left\langle \mathbf{F}(\boldsymbol{y}), \mathbf{V}_j(\boldsymbol{y}, \boldsymbol{p})\right\rangle d\boldsymbol{y}\, d\boldsymbol{p},$$

where $\mathbf{F}$ is any vector field in $C_0^\infty(\Omega; \mathbb{C}^3)$.

Proof. Let $\mathbf{F} \in C_0^\infty(\Omega; \mathbb{C}^3)$, arbitrary. Denote by $s(\nu; \epsilon)$ the square root of $\nu - i\epsilon$ lying in the upper half-plane of $\mathbb{C}$.

With this notation, Stone's formula and Proposition 4.4 imply:

$$\left\|E_\lambda(\mathbf{L})\mathbf{F}\right\|^2_{L^2(\Omega;\mathbb{C}^3)} = \lim_{\epsilon \to 0^+} \frac{\epsilon}{\pi} \int_0^\lambda \left\|R_{\mathbf{L}}(\nu - i\epsilon)\mathbf{F}\right\|^2_{L^2(\Omega;\mathbb{C}^3)} d\nu$$

$$= \lim_{\epsilon\to 0^+} \frac{\epsilon}{\pi} \int_0^\lambda \left\| \mathsf{B}_{\mathbf{F}}\big(\cdot\,; s(\nu;\epsilon)\big) \right\|^2_{\bigoplus^3 L^2(\mathbb{R}^3)} d\nu, \tag{19}$$

where $\mathsf{B}_{\mathbf{F}}$ is as in (16).

We observe that

$$\left| B^{(j)}_{\mathbf{F}}\big(\boldsymbol{p}; s(\nu;\epsilon)\big)\right|^2 = \left(\frac{c_j}{2\pi}\right)^3 \frac{1}{(|p|^2-\nu)^2+\epsilon^2} \left| \int_\Omega \big\langle \mathbf{F}(\boldsymbol{y}), \mathbf{W}_j\big(\boldsymbol{y}, \boldsymbol{p}; s(\nu;\epsilon)\big)\big\rangle\, d\boldsymbol{y} \right|^2. \tag{20}$$

For every fixed $\boldsymbol{y}$, the vector field $\mathbf{W}_j(\boldsymbol{y}, \boldsymbol{p}; s)$ is jointly continuous with respect to $\boldsymbol{p}$ and s; also (e.g., [22, Chapter 8]):

$$\lim_{\epsilon\to 0} \mathbf{W}_j\big(\boldsymbol{y}, \boldsymbol{p}; s\big(|\boldsymbol{p}|^2;\epsilon\big)\big) = \mathbf{V}_j(\boldsymbol{y}, \boldsymbol{p}). \tag{21}$$

Let

$$\mathsf{g}^{(j)}_\epsilon(\boldsymbol{p};\nu) = \int_\Omega \big\langle \mathbf{F}(\boldsymbol{y}), \mathbf{W}_j\big(\boldsymbol{y}, \boldsymbol{p}; s(\nu;\epsilon)\big)\big\rangle\, d\boldsymbol{y},$$

$$\mathsf{g}^{(j)}(\boldsymbol{p}) = \int_\Omega \big\langle \mathbf{F}(\boldsymbol{y}), \mathbf{V}_j(\boldsymbol{y}, \boldsymbol{p})\big\rangle\, d\boldsymbol{y}.$$

Since $\mathbf{F}$ has compact support, relation (21) implies that

$$\lim_{\epsilon\to 0^+} \mathsf{g}^{(j)}_\epsilon\big(\boldsymbol{p}; |\boldsymbol{p}|^2\big) = \mathsf{g}^{(j)}(\boldsymbol{p}). \tag{22}$$

Then we obtain, for $0 < |\boldsymbol{p}|^2 < \lambda$, the equality

$$\lim_{\epsilon\to 0^+} \int_0^\lambda \frac{\epsilon}{\pi[(|\boldsymbol{p}|^2-\nu)^2+\epsilon^2]} \mathsf{g}^{(j)}_\epsilon(\boldsymbol{p};\nu)\, d\nu = \mathsf{g}^{(j)}(\boldsymbol{p}). \tag{23}$$

Here, we have used the fact that, in the sense of distributions (e.g., [21, Theorem 1.17]):

$$\lim_{\epsilon\to 0} \frac{\epsilon}{\pi} \frac{1}{(|\boldsymbol{p}|^2-\nu)^2+\epsilon^2} = \delta\big(|\boldsymbol{p}|^2-\nu\big). \tag{24}$$

By (19), (20) and Fubini's theorem it follows that

$$\begin{aligned}
\left\| E_\lambda(\mathbf{L})\mathbf{F} \right\|^2_{L^2(\Omega;\mathbb{C}^3)} &= \lim_{\epsilon\to 0^+} \frac{\epsilon}{\pi} \int_0^\lambda \left\| \mathsf{B}_{\mathbf{F}}\big(\cdot\,; s(\nu;\epsilon)\big) \right\|^2_{\bigoplus^3 L^2(\mathbb{R}^3)} d\nu \\
&= \sum_{j=1}^3 \left(\frac{c_j}{2\pi}\right)^3 \lim_{\epsilon\to 0^+} \frac{\epsilon}{\pi} \int_0^\lambda \left\| B^{(j)}_{\mathbf{F}}\big(\cdot\,; s(\nu;\epsilon)\big) \right\|^2_{L^2(\mathbb{R}^3)} d\nu \\
&= \sum_{j=1}^3 \left(\frac{c_j}{2\pi}\right)^3 \lim_{\epsilon\to 0^+} \int_{\mathbb{R}^3} \int_0^\lambda \frac{\epsilon\, \mathsf{g}^{(j)}_\epsilon(\boldsymbol{p};\nu)}{\pi[(|\boldsymbol{p}|^2-\nu)^2+\epsilon^2]}\, d\nu\, d\boldsymbol{p}.
\end{aligned}$$

From formula (23) and the fact that $\mathsf{g}^{(j)}$ is in the Schwarz class of functions with rapid decrease at infinity, we can apply the dominated convergence theorem in order to obtain:

$$\begin{aligned}\|E_\lambda(\mathbf{L})\mathbf{F}\|^2_{L^2(\Omega;\mathbb{C}^3)} &= \sum_{j=1}^{3}\left(\frac{c_j}{2\pi}\right)^3 \int_{\mathbb{R}^3} \lim_{\epsilon\to 0^+}\int_0^\lambda \frac{\epsilon\, \mathsf{g}^{(j)}_\epsilon(\boldsymbol{p};\nu)}{\pi[(|\boldsymbol{p}|^2-\nu)^2+\epsilon^2]}\,d\nu\,d\boldsymbol{p} \\ &= \sum_{j=1}^{3}\left(\frac{c_j}{2\pi}\right)^3 \int_{B(\sqrt{\lambda})} \left|\mathsf{g}^{(j)}(\boldsymbol{p})\right| d\boldsymbol{p}. \end{aligned} \tag{25}$$

Now, we define an operator

$$\begin{aligned}&\mathcal{U}: L^2(\mathbb{R}^3;\mathbb{C}^3) \to \bigoplus^3 L^2(\mathbb{R}^3) \\ &\text{by}\quad (\mathcal{U}\mathbf{F})_j(\boldsymbol{p}) = \left(\frac{c_j}{2\pi}\right)^3 \int_\Omega \langle \mathbf{F}(\boldsymbol{y}), \mathbf{V}_j(\boldsymbol{y},\boldsymbol{p})\rangle\, d\boldsymbol{y}.\end{aligned} \tag{26}$$

The operator $\mathcal{U}$ thus defined is an isometry from $L^2(\Omega;\mathbb{C}^3)$ into the space $\bigoplus^3 L^2(\mathbb{R}^3)$, as follows by making $\lambda\to\infty$ in (25). With this definition, formula (25) becomes

$$\|E_\lambda(\mathbf{L})\mathbf{F}\|^2_{L^2(\Omega;\mathbb{C}^3)} = \sum_{j=1}^{3}\int_{B(\sqrt{\lambda})} \left|(\mathcal{U}\mathbf{F})_j(\boldsymbol{p})\right|^2 d\boldsymbol{p}.$$

By polarization we can recover the inner product for arbitrary $\mathbf{F},\mathbf{G}\in C_0^\infty(\Omega;\mathbb{C}^3)$, obtaining:

$$(E_\lambda\mathbf{F},\mathbf{G})_{L^2(\Omega;\mathbb{C}^3)} = (E_\lambda\mathbf{F}, E_\lambda\mathbf{G})_{L^2(\Omega;\mathbb{C}^3)} = \sum_{j=1}^{3}\int_{B(\sqrt{\lambda})} (\mathcal{U}\mathbf{F})_j(\boldsymbol{p})\overline{(\mathcal{U}\mathbf{G})_j(\boldsymbol{p})}\,d\boldsymbol{p}.$$

Substituting $(\mathcal{U}\mathbf{G})_j$ by its definition yields

$$(E_\lambda\mathbf{F},\mathbf{G})_{L^2(\Omega;\mathbb{C}^3)} = \sum_{j=1}^{3}\int_{B(\sqrt{\lambda})} (\mathcal{U}\mathbf{F})_j(\boldsymbol{p})\int_\Omega \langle\overline{\mathbf{G}(\boldsymbol{y}),\mathbf{V}_j(\boldsymbol{y},\boldsymbol{p})}\rangle\,d\boldsymbol{y}\,d\boldsymbol{p}.$$

Since $\mathbf{G}$ has compact support, we can switch the order of integration, and it follows from the arbitrariness of $\mathbf{G}\in C_0^\infty(\Omega;\mathbb{C}^3)$ that

$$(E_\lambda\mathbf{F})(\boldsymbol{y}) = \sum_{j=1}^{3}\int_{B(\sqrt{\lambda})} \mathbf{V}_j(\boldsymbol{y},\boldsymbol{p})(\mathcal{U}\mathbf{F})_j(\boldsymbol{p})\,d\boldsymbol{p}.$$

Writing in the definition of $\mathcal{U}$, we obtain the result stated. $\square$

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